MTH 201
Multivariable calculus and differential equations
Homework 1
Vectors, dot product, and cross product

Notation: $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ donote vectors in $\mathbb{R}^3$, unless specified.

1. Let $\mathbf{a} = \langle a_1, a_2 \rangle$ be a vector in $\mathbb{R}^2$. Compute $2\mathbf{a}$, $\frac{1}{2}\mathbf{a}$, and $-3\mathbf{a}$. Graph all four vectors.

2. Given a non-zero vector $\mathbf{v}$, find a unit vector that points in the same direction as $\mathbf{v}$.

3. Determine if vectors in the following sets are parallel
   (a) $\mathbf{u} = \langle -2, 4, -1 \rangle,\ \mathbf{v} = \langle 6, -12, 3 \rangle$
   (b) $\mathbf{u} = \langle 2, 5 \rangle,\ \mathbf{v} = \langle 2, 8 \rangle$.

4. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be position vectors of points P_1 and P_2 respectively. Use vector algebra (vector addition and scalar multiplication) to find the midpoint of the line segment joining P_1 and P_2.

5. Let $\mathbf{u} = \langle 6, 3, 2 \rangle$ and $\mathbf{v} = \langle 1, -2, -2 \rangle$ be two vectors.
   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ − the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

6. Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero vectors. Then
   (a) under what circumstances, the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal?
   (b) show that $|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2)$.

7. Use vectors to prove that the diagonals of a parallelogram bisect each other. (HW)

8. Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be non-zero vectors. Then show that the vector $\mathbf{w} = |\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}$ bijects the angle between $\mathbf{u}$ and $\mathbf{v}$.

9. Under what assumption on vectors, we get equality in Cauchy-Schwartz inequality?

10. If $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, then prove that
$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

11. Suppose $\mathbf{u} \neq 0$,
   (a) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, does it follow that $\mathbf{v} = \mathbf{w}$.
   (b) If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, does it follow that $\mathbf{v} = \mathbf{w}$.
   (c) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, does it follow that $\mathbf{v} = \mathbf{w}$.

12. Given three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$, show that
   (a) $|\mathbf{u} - \mathbf{v}| \leq |\mathbf{u} - \mathbf{w}| + |\mathbf{v} - \mathbf{w}|$.
   (b) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$. (HW)
   (c) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$.

13. Find volume of the parallelopiped determined by vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + 3\mathbf{k}$, and $7\mathbf{j} - 4\mathbf{k}$.

14. Which of the followings are meaningful? Which are meaningless? Justify your answers. (HW)

    (a) $\mathbf{u} + (\mathbf{v} \cdot \mathbf{w})$

    (b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$

    (c) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

    (d) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

    (e) $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$